Progressive Measures

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Definition 0.1 (Kra et al., 2025, Definition 3.1)**.** Let (X, T) be a topological system and $k \in \mathbb{N}$. We say that a probability measure $\tau \in \mathcal{M}(X^{k+1})$ is *progressive* if, for all open sets $U_1, ..., U_k \subset X$ with

$$\tau(X \times U_1 \times \cdots \times U_k) > 0$$

there exist infinitely many $n \in \mathbb{N}$ such that

$$\tau((X \times U_1 \times \cdots \times U_{k-1} \times U_k) \cap T_{\triangle}^{-n}(U_1 \times U_2 \times \cdots \times U_k \times X)) > 0.$$

Definition 0.2 (cf. Kra et al., 2025, Definition 3.1)**.** Let (X, Γ) be a topological system and $k \in \mathbb{N}$. We say that a probability measure $\tau \in \mathcal{M}(X^{k+1})$ is *progressive* if, for all open sets $U_1, ..., U_k \subset X$ with

$$\tau(X \times U_1 \times \cdots \times U_k) > 0$$

there exist infinitely many $\gamma \in \Gamma$ such that

$$\tau((X \times U_1 \times \cdots \times U_{k-1} \times U_k) \cap (T\gamma) - 1.U_1 \times U_2 \times \cdots \times U_k \times X) > 0.$$

If $(a, x_1, ..., x_k)$ is a $k+1$-term Erdős progression where U_j is a neighbourhood of x_j for $1 \leq j \leq k$, then

$$B = (X \times U_1 \times \cdots \times U_{k-1} \times U_k) \cap T_{\triangle}^{-n}(U_1 \times U_2 \times \cdots \times U_k \times X)$$

has $(a, x_1, ..., x_k) \in B$ and, if further $\tau(B) > 0$, B contains infinitely many other $k+1$-term Erdős progressions.

Similarly, if $(a, x_{0...01}, ..., x_{1...1})$ is a k-dimensional Erdős cube where U_j is a neighbourhood of x_j for all $j \in \{0,1\}^k$, $c_i : \{0,1\}^k \to \{0,1\}^k$ such that

$$c_i(j)(n) = \begin{cases} 1 & j = n, \\ j(n) & \text{otherwise,} \end{cases}$$

and $C__, then

$$B = (X \times U_1 \times \cdots \times U_{k-1} \times U_k) \cap T_{\triangle}^{-n}(U_1 \times U_2 \times \cdots \times U_k \times X)$$

Definition 0.3 (cf. Kra et al., 2025, Definition 3.1)**.** Let (X, Γ) be a topological system and $k \in \mathbb{N}$. We say that a probability measure $\tau \in \mathcal{M}(X^{k+1})$ is *progressive* if, for all open sets $U_1, ..., U_k \subset X$ with

$$\tau(X \times U_1 \times \cdots \times U_k) > 0$$

there exist infinitely many $\gamma \in \Gamma$ such that

$$\tau((X \times U_1 \times \cdots \times U_{k-1} \times U_k) \cap T^{\gamma} U_1 \times U_2 \times \cdots \times U_k \times X) > 0.$$

If $(a, x_1, ..., x_k)$ is a $k+1$-term Erdős progression where U_j is a neighbourhood of x_j for $1 \leq j \leq k$, then

$$B = (X \times U_1 \times \cdots \times U_{k-1} \times U_k) \cap T_{\triangle}^{-n}(U_1 \times U_2 \times \cdots \times U_k \times X)$$

has $(a, x_1, ..., x_k) \in B$ and, if further $\tau(B) > 0$, B contains infinitely many other $k+1$-term Erdős progressions.

Proposition 0.1 (Kra et al., 2025, Proposition 3.2)**.** *Fix $k \in \mathbb{N}$ and let (X, T) be a topological system. Let $a \in X$, $U_1, ..., U_k \subset X$ be open sets, and let $\tau \in \mathcal{M}(X^{k+1})$ be a progressive measure satisfying $\tau(\{a\} \times X^k) = 1$.*

If $\tau(X \times U_1 \times \cdots \times U_k) > 0$, then there exists an Erdős progression $(a, x_1, ..., x_k)$ with $x_j \in U_j$ for all $1 \leq j \leq k$.

Proof. Let $V = U_1 \times \cdots \times U_k$. As τ is a progressive measure, we can find $c(1) \in \mathbb{N}$ such that

$$\tau((X \times V) \cap T_{\triangle}^{-c(1)}(V \times X)) > 0.$$

□

1 Constructing Progressive Measures

1. Check that the system has topological pronilfactors.
2. Define a product measure in the $k-1$-step pronilfactor that is generic with Φ and $a \in \text{gen}(\mu, \Phi)$ and structured similarly to the progression.
3. Project this measure back to the system by using conditional expectations.

2 Claim seemingly related to Progressive Measures

We will look at the following result to see how progressive measures are involved and determine whether we can generalise this:

Lemma 2.1 (Kra et al., 2026, Claim 1)**.** *Let (X, μ, T) be an ergodic measure-preserving system where every measurable eigenfunction is equal μ-almost everywhere to a topological eigenfunction.Also let λ be a self-joining of (X, μ, T),*

$d : X \times X \to [0, \infty)$ denote a metric on X, and $(x, y) \in gen(\lambda, \Phi) \cap (gen(\mu, \Phi) \times gen(\mu, \Phi)) \cap (X \times supp\ \mu)$ for a Følner sequence Φ.

For any clopen set $E \subset X$ with $\mu(E) > 0$ and any measurable set $Y \subset X \times X$: If $\lambda((X \times E) \cap Y) > 0$ then, for every $\varepsilon > 0$, there exist infinitely many $\gamma \in \Gamma$ such that

$$d(T^b x, y) < \varepsilon \qquad \text{and} \qquad \lambda\left((T^{-b}X \times E) \cap Y\right) > 0.$$

The proof of this result uses the following results:

...

We follow and breakdown the proof of Lemma 2.1 given by Kra et al. (2026):

Proof. Let $f_{\mathrm{c}} = \mathbb{E}(1_E \mid \mathscr{K})$ and $f_{\mathrm{wm}} = 1_E - f_{\mathrm{c}}$. As

$$\begin{aligned}
\mathbb{E}(f_{\mathrm{wm}} \mid \mathscr{K}) &= \mathbb{E}(1_E - \mathbb{E}(1_E \mid \mathscr{K}) \mid \mathscr{K}) \\
&= \mathbb{E}(1_E \mid \mathscr{K}) - \mathbb{E}(1_E \mid \mathscr{K}) \\
&= 0,
\end{aligned}$$

then, by **?@thm-WeakMixing2**, f_{wm} is mixing.

By **?@prp-EigenAlgebraProp2**, $f_{\mathrm{c}} > 0$ for μ-almost every $x \in E$. This gives us

$$\begin{aligned}
0 < \lambda((X \times E) \cap Y) &= \int (1 \otimes 1_E) \cdot 1_Y \, \mathrm{d}\lambda \\
&= \int (1 \otimes f_{\mathrm{c}}) \cdot 1_Y \, \mathrm{d}\lambda \\
&= \eta,
\end{aligned}$$

where $\eta = \langle (1 \otimes f_{\mathrm{c}}), 1_Y \rangle$.

> Though Kra et al. (2026) focuses on $b \in \mathbb{N}$ from this point on, we would need to generalise to an amenable group Γ.

As $f_{\mathrm{c}} \in \mathrm{L}^2(X, \mathscr{K}, \mu)$ and that $\mathrm{L}^2(X, \mathscr{K}, \mu)$ is defined to be the closed subspace of $\mathrm{L}^2(X, \mathrm{Bor}(X), \mu)$ spanned by its measurable eigenfunctions, then f_{c} can be approximated by eigenfunctions. By the assumption that every measurable eigenfunction in (X, μ, T) is equal μ-almost everywhere to a topological eigenfunction, there exists $r \in \mathbb{N}$ and topological eigenfunctions $\chi_1, ..., \chi_r \in \mathscr{C}(()X)$ such that $f'_{\mathrm{c}} = \chi_1 + \cdots + \chi_r$ satisfies $||f'_{\mathrm{c}} - f_{\mathrm{c}}|| < \eta/6$. Note, by construction, $f'_{\mathrm{c}} \in \mathscr{C}(()X)$ too.

Let $\delta > 0$ be sufficiently small such that, for $1 \leq i \leq r$ and for all $z \in B_\delta(y)$, we have $|\chi_i(y) - \chi_i(z)| < \eta/6r$. As χ_i is a topological eigenfunction, we also have ☐

- Will have to look at finite dimensional representations for Amenable Groups setting instead of eigenfunctions

  - For actions of abelian groups, the Kronecker factor is spanned by eigenfunctions. For actions of general amenable groups, the Kronecker factor is made up of finite-dimensional representations.
  - A finite dimensional representation is a homomorphism from G to U(n) the group of nxn unitary matrices.
    - When n = 1 this is just a homomorphism to the circle.
    - When also G = Z this is just the choice of an eigenvalue.
  - What we are used to is that the Kronecker factor is made up of eigenfunctions.
    - More generally, the Kronecker factor is made up of spaces that look like K/L where K is a compact group and L is a closed subgroup. The action of Gamma on each such space will be determined by a homomorphism alpha from G to K. Thus the group element g in Gamma acts on the coset kL by alpha(g)kL. This is an action by isometries because we can always fix an invariant metric on K/L. This is enough to generalize the sentence in the claim that you mentioned during our meeting.
    - In this more general world, instead of eigenfunctions chi, we have "matrix coefficients" which are functions of the form chi(x) = < pi(x) u, v > where u and v are vectors in L2(K/L).
- Could use the tensor product representation with its dual representation?
  - Show $f_c \otimes \bar{f}_c \in \mathrm{fix}(U \otimes \bar{U})$.
    - Or construct f_c such that this is the case.
    - Or write f_c in terms of its orthonormal bases as it belongs to an irreducible invariant finite-dimensional subspace M. If so, T^g would appear to be a matrix transformation A_g on M that preserves the inner product. The matrix coefficients will be defined by $(T^g e_i \mid e_j)$. A_g is a unitary matrix to conserve measures and $g \mapsto A_g$ is a homomorphism mapping from G to a compact group of unitary matrices (prove).
  - Define topological function analogue, using

$$\sum_{i=1}^{n} e_i \otimes \bar{e}_i,$$

    where $\{e_1, ..., e_n\}$ is an orthonormal basis of an invariant finite-dimensional subspace $M \subseteq H$.
  - Use the fact that the linear hull of these topological function analogues are dense in $\mathrm{fix}(U \otimes \bar{U})$ to construct a topological function analogue that is "close" to f_c.

3 Discrete Spectrum & Eigenmatrices

Definition 3.1 (cf. Jamneshan and Kreidler, 2025, Definition 5.1.2)**.** For a unitary representation $U : G \to \mathscr{U}(\mathcal{L}(X, \mathrm{Bor}(X)))$, we call an invariant finite-dimensional subspace $M \subseteq \mathscr{L}(X, \mathrm{Bor}(X))$ *irreducible* if $\{0\}$ and M are the only invariant linear subspaces contained in M.

Definition 3.2 (cf. Jamneshan and Kreidler, 2025, Definition 5.1.1 & Corollary 5.1.5)**.** Let $U : G \to \mathscr{U}(\mathscr{L}(X))$ be a unitary representation of a group G. Then the closure

$$\mathscr{L}(X, \mathrm{Bor}(X))_{\mathrm{ds}} := \overline{\mathrm{lin}} \bigcup \{M \subseteq \mathscr{L}(X, \mathrm{Bor}(X)) \mid M \text{ irreducible invariant finite-dimensional subspace}\} \subseteq \mathscr{L}(X)$$

is called the *discrete spectrum part* of U. As $M_1 + M_2$ is an invariant finite-dimensional subspace for invariant finite-dimensional subspaces M_1, M_2, then $\mathscr{L}(X)$ds is always a (closed) linear subspace of $\mathscr{L}(X)$.

As there exists $\pi : X \to K/L$ where K is a compact group, $L \subseteq K$ is a closed subgroup and m is an invariant measure on K/L such that $\mathrm{L}^2(X, \mathscr{K}, \mu) \cong \mathrm{L}^2(K/L, m)$ and, for $f : K/L \to \mathbb{C}$, $f \circ \pi : X \to \mathbb{C}$.

As $\mathrm{L}^2(X, \mathscr{K}, \mu) \cong \mathrm{L}^2(K/L, m)$, we know that π^{-1} exists and thus there exists $\bar{f}_{\mathrm{c}} \in \mathrm{L}^2(K/L, m)$ such that $\bar{f}_{\mathrm{c}} = f_{\mathrm{c}} \circ \pi^{-1}$.

Theorem 3.1 (Farashahi, 2017, Theorem 4.6)**.** *Let H be a closed subgroup of a compact group G and μ be the normalized G-invariant measure on G/H. The Hilbert space $L^2(G/H, \mu)$ satisfies the following orthogonality decomposition*

$$L^2(G/H, \mu) = \bigoplus_{[\pi] \in G/H^*} \epsilon_\pi(G/H),$$

where ϵ_π denotes the linear span of the matrix elements of π. (π may not be the same as the previous one)

Note:

- (Farashahi, 2017, Theorem 4.5) will be helpful in confirming that every continuous function can be approximated equicontinuously with a linear combination of "eigenfunctions".

Let $\{e_1, ..., e_n\}$ be the (continuous) orthonormal basis of a sufficiently large invariant finite-dimensional subspace $M \subseteq \mathrm{L}^2(X, \mathscr{K}, \mu)$ such that there exists $h \in M$ where $||f_{\mathrm{c}} - h||_{\mathrm{L}^2} < \eta/6$ and

$$h = \sum_{i=1}^{n} \langle h, e_i \rangle e_i.$$

By the Cauchy-Schwartz Inequality,

$$\langle h, e_i \rangle \leq ||h||_{\mathrm{L}^2} \cdot ||e_i||_{\mathrm{L}^2} = ||h||_{\mathrm{L}^2}.$$

We also have

$$\begin{aligned}
U_g h &= \sum_{i=1}^{n} \langle h, e_i \rangle \sum_{j=1}^{n} \langle U_g e_i, e_j \rangle e_j \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \langle h, e_i \rangle \langle U_g e_i, e_j \rangle \right) e_j.
\end{aligned}$$

As we have

$$\langle U_g e_i, e_j \rangle = \langle e_i, U_g^{-1} e_j \rangle,$$

then

$$\begin{aligned}
U_g h &= \sum_{j=1}^{n} \left(\sum_{i=1}^{n} \langle h, e_i \rangle \langle U_g e_i, e_j \rangle \right) e_j \\
&= \sum_{j=1}^{n} \left\langle \sum_{i=1}^{n} \langle h, e_i \rangle e_i, U_g^{-1} e_j \right\rangle e_j \\
&= \sum_{j=1}^{n} \langle h, U_g^{-1} e_j \rangle e_j.
\end{aligned}$$

Let $\delta > 0$ be sufficiently small such that, for $1 \leq i \leq n$, we have $|e_i(y) - e_i(z)| < \eta/6n$ whenever $d(z, y) < \delta$. Applying U_g and focusing on each $1 \leq i \leq n$, we want to show that

$$|U_g e_i(y) - U_g e_i(z)| < \eta/6n$$

for all $g \in G$.

We have

$$U_g e_i = \sum_{j=1}^{n} \langle U_g e_i, e_j \rangle e_j.$$

For each e_j, we have that

- Use continuity as e_j is assumed to be continuous
- Orthogonal invariance
- Kronecker factor is uniquely ergodic so every point has dense orbit

Below is incorrect at the moment:

, as $e_1, ..., e_n$ is an orthonormal basis and $|e_j(y) - e_j(z)| < \eta/6n$ for all $1 \leq j \leq n$, we get

$$\begin{aligned}
|U_g e_i(y) - U_g e_i(z)|^2 &= \sum_{j=1}^{n} |\langle U_g e_i, e_j \rangle (e_j(y) - e_j(z))|^2 \\
&= \left(\frac{\eta}{6n} \right)^2 \sum_{j=1}^{n} |\langle U_g e_i, e_j \rangle|^2 \\
&= \left(\frac{\eta}{6n} \right)^2.
\end{aligned}$$

Thus,

$$|U_g e_i(y) - U_g e_i(z)| < \frac{\eta}{6n}.$$

3.1 Rough work

By definition of the dual representation, we have

$$\begin{aligned}
\overline{h}(f) &= \langle f, h\rangle \\
&= \langle f, \sum_{i=1}^{n} \langle h, e_i\rangle e_i\rangle \\
&= \sum_{i=1}^{n} \overline{\langle h, e_i\rangle} \langle f, e_i\rangle \\
&= \sum_{i=1}^{n} \overline{\langle h, e_i\rangle}\, \overline{e_i}(f),
\end{aligned}$$

for any $f \in \mathcal{L}(X)$. Thus,

$$\overline{h} = \sum_{i=1}^{n} \overline{\langle h, e_i\rangle}\, \overline{e_i}.$$

With our tensor product $U \otimes \overline{U} : G \to \mathscr{U}(\mathcal{L}(X) \otimes \mathcal{L}(\mathcal{L}(X)))$ with the dual representation, we get

$$\begin{aligned}
(U_g \otimes \overline{U_g})(h \otimes \overline{h}) &= (U_g h) \otimes (\overline{U_g h}) \\
&= \sum_{i=1}^{n} \left(\langle h, e_i\rangle U_g e_i\right) \otimes \left(\overline{\langle h, e_i\rangle}\, \overline{U_g e_i}\right) \\
&= \sum_{i=1}^{n} \left(\langle h, e_i\rangle \sum_{j=1}^{n} \langle U_g e_i, e_j\rangle e_j\right) \otimes \left(\overline{\langle h, e_i\rangle} \sum_{k=1}^{n} \overline{\langle U_g e_i, e_k\rangle}\, \overline{e_k}\right)
\end{aligned}$$

for all $g \in G$.

Continuing on, we have

$$\begin{aligned}
(U_g \otimes \overline{U_g})(h \otimes \overline{h}) &= \sum_{i=1}^{n} \left(\langle h, e_i\rangle \sum_{j=1}^{n} \langle U_g e_i, e_j\rangle e_j\right) \otimes \left(\overline{\langle h, e_i\rangle} \sum_{k=1}^{n} \overline{\langle U_g e_i, e_k\rangle}\, \overline{e_k}\right) \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} \left(\sum_{i=1}^{n} \langle h, e_i\rangle \overline{\langle h, e_i\rangle} \langle U_g e_i, e_j\rangle \overline{\langle U_g e_i, e_k\rangle}\right) e_j \otimes \overline{e_k}.
\end{aligned}$$

As U_x is unitary, then $U_g e_1, ..., U_g e_n$ also forms an orthonormal basis and we get

$$e_k = \sum_{i=1}^{n} \langle e_k, U_g e_i\rangle U_g e_i.$$

We also have

$$\sum_{i=1}^{n} \langle h, e_i\rangle \overline{\langle h, e_i\rangle} \langle U_g e_i, e_j\rangle \overline{\langle U_g e_i, e_k\rangle} = \sum_{i=1}^{n} \langle \langle e_i, h\rangle h, e_i\rangle \langle \langle e_k, U_g e_i\rangle U_g e_i, e_j\rangle$$

As

$$U_g h = \sum_{i=1}^{n} \langle h, e_i \rangle U_g e_i = \sum_{j=1}^{n} \langle h, U_g^{-1} e_j \rangle e_j$$

and $U_g e_1, ..., U_g e_n$ are orthonormal, we get

$$\langle h, e_i \rangle U_g e_i = \sum_{j=1}^{n} \langle h, U_g^{-1} e_j \rangle \langle e_j, U_g e_i \rangle U_g e_i$$

Using the Cauchy-Schwartz Inequality, we get

$$\begin{aligned} |\langle h, U_g^{-1} e_i \rangle| &\leq ||h|| \cdot ||U_g^{-1} e_i|| \\ &= ||h|| \cdot ||e_i|| \\ &= ||h|| \end{aligned}$$

We have

$$U_g e_i = \sum_{j=1}^{n} \langle U_g e_i, e_j \rangle e_j,$$

and so

$$U_g e_i(y) - U_g e_i(z) = \sum_{j=1}^{n} \langle U_g e_i, e_j \rangle (e_j(y) - e_j(z)).$$

As $|e_j(y) - e_j(z)| < \eta/6n$ for all $1 \leq j \leq n$, we get

$$\begin{aligned} |U_g e_i(y) - U_g e_i(z)| &\leq \sum_{j=1}^{n} \langle U_g e_i, e_j \rangle |(e_j(y) - e_j(z))| \\ &< \frac{\eta}{6n} \sum_{j=1}^{n} \langle U_g e_i, e_j \rangle \end{aligned}$$

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