Erdős Cubes

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Unless specified otherwise, let (X, μ, Γ) be the Furstenberg system, Ψ a Følner sequence and a be a point in X defined by the countably-infinite discrete amenable group Γ, the Følner sequence Φ, and the subset $A \subset \Gamma$ where $A = \{\gamma \in \Gamma : T^\gamma a \in E\}$. Additionally, let $[[k]] = \{0,1\}^k$ for some $k \in \mathbb{N}$ and define $X^{[[k]]}$ to be $\prod_{i=1}^{2^k} X$ and $T^{[[k]]}$ to be the corresponding 2^k-fold transformation on $X^{[[k]]}$.

We looked at "Infinite Sumsets in Sets with Positive Density" (Kra et al., 2022b) in our goal of understanding Erdős cubes for the purpose of generalising the paper by Host (2019) using more recent methods.

Definition 0.1 (Kra et al., 2022b, Definition 1.2)**.** We call $(x_{00}, x_{01}, x_{10}, x_{11}) \in X^{[[2]]}$ a *2-dimensional Erdős cube* if there exists sequences $n \mapsto b(n), c(n)$ of distinct elements in Γ such that

$$(T \times T)^{c_1(n)}(x_{00}, x_{01}) \to (x_{10}, x_{11}),$$
$$(T \times T)^{c_2(n)}(x_{00}, x_{10}) \to (x_{01}, x_{11}).$$

Whilst understanding Erdős cubes, we also briefly compared it to Erdős progressions to understand how these similar concepts differ.

Definition 0.2 (Kra et al., 2022a, Definition 2.1)**.** We call $(x_0, x_1, x_2) \in X^3$ a *3-term Erdős progression* if there exists a sequence $n \mapsto c(n)$ of distinct elements in Γ such that

$$(T \times T)^{c(n)}(x_0, x_1) \to (x_1, x_2).$$

We then worked through a basic example of showing that there exists an Erdős cube in a Furstenberg dynamical system if and only if there exist infinite sumsets within a positively dense set in a group.

Theorem 0.1. *The following statements are equivalent:*

1. *There exists infinite sets $B_1, B_2 \subset \Gamma$ where $B_1 \cdot B_2 \subset A$.*
2. *There exists an Erdős cube $(a, x_{01}, x_{10}, x_{11}) \in X^{[[2]]}$ where $x_{11} \in E$.*

Proof.

(1) $\Rightarrow$ (2): Assume we have infinite sets $B_1, B_2 \subset \Gamma$ where

$$B_1 \cdot B_2 \subset A = \{\gamma \in \Gamma : T^g a \in E\}.$$

Let $n \mapsto b_1(n)$ and $m \mapsto b_2(m)$ be infinite sequences of distinct elements in B_1 and B_2, respectively. As $b_1(n) \cdot b_2(m) \in A$ for all $n, m \in \mathbb{N}$, we have

$$T^{b_1(n) \cdot b_2(m)} a \in E$$

for all $n, m \in \mathbb{N}$. As X is compact, we use the property that every sequence has a convergent subsequence to define subsequences $(c_1(n))_{n \in \mathbb{N}}$ and $(c_2(m))_{m \in \mathbb{N}}$ such that

$$\lim_{n \to \infty} T^{c_1(n)} a,$$
$$\lim_{m \to \infty} T^{c_2(m)} a,$$

and

$$\lim_{m \to \infty} T^{c_1(n) \cdot c_2(m)} a$$

exist and label these limits as x_{10}, x_{01}, and x_{11}, respectively. As $T^{c_1(n) \cdot c_2(m)} a \in E$ for all $n, m \in \mathbb{N}$, we have $x_{11} \in E$. Thus, we have shown the existence of an Erdős cube where $x_{11} \in E$.

(2) $\Rightarrow$ (1): Assume we have an Erdős cube $(a, x_{01}, x_{10}, x_{11}) \in X^{[[2]]}$ where $x_{11} \in E$. We construct subsequences $n \mapsto b_1(n)$ and $m \mapsto b_2(m)$ inductively. We start by choosing $b_1(1)$ from $\{c_1(n) : n \in \mathbb{N}\}$ such that

$$\lim_{m \to \infty} T^{b_1(1) \cdot c_2(m)} a \in E.$$

Then, we choose $b_2(1)$ from $\{c_2(m) : m \in \mathbb{N}\}$ such that

$$\lim_{n \to \infty} T^{c_1(n) \cdot b_2(1)} a \in E.$$

Next, we choose $b_1(2) \neq b_1(1)$ from $\{c_1(n) : n \in \mathbb{N}\}$ such that $b_1(2) \cdot b_2(1) \in \{\gamma \in \Gamma : T^g a \in E\} = A$ and

$$\lim_{m \to \infty} T^{b_1(2) \cdot c_2(m)} a \in E.$$

We then choose $b_2(2) \neq b_2(1)$ from $\{c_2(m) : m \in \mathbb{N}\}$ such that $b_1(1) \cdot b_2(2),\ b_1(2) \cdot b_2(2) \in A$ and

$$\lim_{n \to \infty} T^{c_1(n) \cdot b_2(2)} a \in E.$$

We repeat this alternating inductive construction by choosing

$$b_1(i) \in \{c_1(n) : n \in \mathbb{N}\} \setminus \{b_1(1), ..., b_1(i-1)\}$$

such that $b_1(i) \cdot \{b_2(m) : m < i\} \subset A$ and

$$\lim_{m \to \infty} T^{b_1(i) \cdot c_2(m)} a \in E,$$

then choosing

$$b_2(i) \in \{c_2(m) : m \in \mathbb{N}\} \setminus \{b_2(1), ..., b_2(i-1)\}$$

such that $\{b_1(n) : n \leq i\} \cdot b_2(i) \subset A$ and

$$\lim_{n \to \infty} T^{c_1(n) \cdot b_2(i)} a \in E.$$

After infinitely many steps, we are left with sequences $(b_1(n))_{n \in \mathbb{N}}$ and $(b_2(m))_{m \in \mathbb{N}}$ of distinct elements in Γ such that $T^{b_1(n) \cdot b_2(m)} a \in E$ for all $n, m \in \mathbb{N}$. As $A = \{\gamma \in \Gamma : T^g a \in E\}$, we can conclude that $b_1(n) \cdot b_2(m) \in A$ for all $n, m \in \mathbb{N}$. By setting $B_1 = \{b_1(n) : n \in \mathbb{N}\}$ and $B_2 = \{b_2(m) : m \in \mathbb{N}\}$, we proven the existence of infinite subsets B_1, B_2 such that $B_1 \cdot B_2 \subset A$.

$\square$

Host, B. (2019). *A short proof of a conjecture of erdös proved by moreira, richter and robertson*. Available at: https://arxiv.org/abs/1904.09952

Kra, B. et al. (2022a). *A proof of erdős's $B + B + t$ conjecture*. Available at: https://arxiv.org/abs/2206.12377

Kra, B. et al. (2022b). *Infinite sumsets in sets with positive density*. Available at: https://arxiv.org/abs/2206.01786