Joinings

Kai Prince

2025-09-05

We want to build our understanding of joinings as they help us create a generalised definition of a measure on product spaces that are behave as we would expect for a multi-dimensional measure. This is important when we start looking at n-dimensional Erdős cubes. In building this understanding, we used the product measure as an example and investigated what property we are interested in that we'd like to see in joinings.

Definition 0.1. The *product measure* on $(X \times Y, \mathscr{B} \otimes \mathscr{C})$ is μ where

$$(\mu)(B \times C) = \mu(B)\nu(C)$$

for $B \in \mathscr{B}, C \in \mathscr{C}$.

The product measure is $T \times S$-invariant, i.e., for all $D \in \mathscr{B} \otimes \mathscr{C}$,

$$(\mu)(D) = (\mu)((T \times S)^{-1}D).$$

Note, if $D = B \times C$, then

$$(T \times S)^{-1}(B \times C) = T^{-1}B \times S^{-1}C.$$

The main property we are interested in relate to how we can relate the measure of a product space to the measures of the component spaces. This leads us to the definition of marginals:

Definition 0.2. Let $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ be coordinate projections of a probability space $(X \times Y, \mathscr{B} \times \mathscr{C}, \eta)$. We call the coordinate-projected measures $\pi_X(\eta)$ and $\pi_Y(\eta)$ the *marginals* of η.

These have the property that

$$(\pi_X(\eta))(B) = \eta(\pi_X^{-1}(B))$$

and

$$(\pi_Y(\eta))(C) = \eta(\pi_Y^{-1}(C)).$$

When looking at the product measure $\eta = \mu$ as our example, we get the marginal μ as

$$\begin{aligned}(\pi_X(\mu))(B) &= (\mu)(\pi_X^{-1}(B)) \\ &= (\mu)(B \times Y) \\ &= \mu(B)\nu(Y) \\ &= \mu(B),\end{aligned}$$

so $\pi_X(\mu) = \mu$ and we get $\pi_Y(\mu) = \nu$ similarly. Hence, when looking at measures on product spaces, we want to conserve this property. Thus, we arrive at the definition of joinings:

Definition 0.3. A *joining* of $(X, \mathscr{B}, \mu, T)$ and $(Y, \mathscr{C}, \nu, S)$ is a measure η on $(X \times Y, \mathscr{B} \otimes \mathscr{C})$ satisfying:

1. η is $T \times S$-invariant,
2. and, η has marginals μ and ν.

A *self-joining* of $(X, \mathscr{B}, \mu, T)$ is a joining of $(X, \mathscr{B}, \mu, T)$ and itself.

The product measure is always a joining and it may sometimes be the only joining for some product spaces. For our research, we're interested in the product space of $(X, \mathscr{B}, \mu)$ with itself, so we will be focusing on self-joinings. We move on to this special case of self-joinings:

Proposition 0.1. *Let*

$$\triangle = \{(x, x) : x \in X\} \subseteq X \times X.$$

Then there is a self-joining η of $(X, \mathscr{B}, \mu, T)$ with $\eta(\triangle) = 1$.

For the product measure, we only get $(\mu)(\triangle) > 0$ if $\mu(\{a\}) > 0$, for atoms $a \in X$, and we get $(\mu)(\triangle) = 1$ if and only if $X = \{*\}$, i.e., X only has a single point.

To show that we will there exists a self-joining of $(X, \mathscr{B}, \mu, T)$, η, such that $\eta(\triangle) = 1$, we define $\phi : X \to X \times X$ by $\phi(x) = (x, x)$ and define the push-forward measure of μ under ϕ as

$$\begin{aligned}\phi\mu(B \times C) &= \mu(\phi^{-1}(B \times C)) \\ &= \mu(\{x \in X : x \in B, x \in C\}) \\ &= \mu(B \cap C).\end{aligned}$$

As $\phi^{-1}(\triangle) = X$ and $\mu(X) = 1$, we arrive at $\phi\mu(\triangle) = 1$ and by setting $\eta = \phi\mu$, we complete the proof.

1 Relatively Independent Joinings

Before we can look at the measures required for Erd"{o}s cubes, we also set out to understand relatively independent joinings.

Definition 1.1. A *relatively independent joining* of $(X, \mathscr{B}, \mu, T)$ is the self joining, ν, of $(X, \mathscr{C}, \mu, T)$ where $\mathscr{C}$ is the σ-subalgebra of T-invariant subsets in $\mathscr{B}$.

We construct an example of a relatively independent joining.

Fix $(X, \mathscr{B}, \mu, T)$ and a σ-subalgebra $\mathscr{C} \subseteq \mathscr{B}$ such that $\mathscr{C}$ is T-invariant. This gives us the following properties for all $f \in \mathrm{L}^2(X, \mathscr{B}, \mu)$:

1. $\mathbb{E}_\mu(f \mid \mathscr{C})$ exists.
2. $\mathbb{E}_\mu(Tf \mid \mathscr{C})$ exists as $Tf = f \circ T$.
3. $T\,\mathbb{E}_\mu(f \mid \mathscr{C}) = \mathbb{E}_\mu(Tf \mid \mathscr{C})$.

This allows us to use $\mathscr{C}$ to define a self-joining ν such that

$$\nu(A \times B) = \int \mathbb{E}_\mu(\mathbb{1}_A \mid \mathscr{C}) \cdot \mathbb{E}_\mu(\mathbb{1}_B \mid \mathscr{C}) \,\mathrm{d}\mu.$$

We know this is a probability measure as $\nu(X \times X) = \int \mathbb{1}_X \cdot \mathbb{1}_X \,\mathrm{d}\mu = \mu(X) = 1$. Let $\pi_1, \pi_2 : X \times X \to X$ be our projections of the first and second coordinates, respectively. Fixing $B \in \mathscr{B}$,

$$\begin{aligned} \nu(\pi_1^{-1}(B)) = \nu(B \times X) &= \int \mathbb{E}_\mu(\mathbb{1}_B \mid \mathscr{C}) \cdot \mathbb{E}_\mu(\mathbb{1}_X \mid \mathscr{C}) \,\mathrm{d}\mu \\ &= \int \mathbb{E}_\mu(\mathbb{1}_B \mid \mathscr{C}) \,\mathrm{d}\mu \\ &= \mu(B), \end{aligned}$$

as $\mathbb{E}_\mu(\mathbb{1}_X \mid \mathscr{C}) = 1$. Thus, we have shown that the marginal $\pi_1(\nu)$ is μ and we also have $\pi_2(\nu) = \mu$ similarly. The proof that ν is $T \times T$-invariant is also trivial.

We also proved: If $(X, \mathscr{B}, \mu, T)$ is ergodic, then the relatively independent joining of $(X, \mathscr{B}, \mu, T)$ is the product measure $\mu \times \mu$, and that this may not be true for the relatively independent joining of $(X \times X, \mathscr{B} \times \mathscr{B}, \mu \times \mu, T \times T)$ as $(X \times X, \mathscr{B} \times \mathscr{B}, \mu \times \mu, T \times T)$ may not be an ergodic system.