Functional Analysis

Kai Prince

2025-09-05

The foundational knowledge relating to Measure & Ergodic Theory, that has not been covered elsewhere, was provided by Jamneshan and Kreidler (2025).

Definition 0.1. For a Hilbert space H,

1. A *linear isometry* is a linear map $U : H \to H$ such that $||Uf|| = ||f||$ for all $f \in H$ (and thus injective).
2. A linear isometry, U, is a *unitary operator* if it is surjective (and thus bijective).
3. We write $\mathscr{U}(H)$ for the set of all unitary operators $U : H \to H$.
4. For a group Γ, we call a group homomorphism $U : \Gamma \to \mathscr{U}(H)$, $\gamma \mapsto U_\gamma$ a *unitary representation* of Γ on H.

Definition 0.2. Let $J : (Y, S) \to (X, T)$ be an extension of measure-preserving systems. Then $E = U_J(\mathrm{L}^2(Y))$ is an *invariant Markov sub-lattice* of $\mathrm{L}^2(X)$, i.e.,

1. E is a closed linear subspace of $\mathrm{L}^2(X)$,
2. $1 \in E$,
3. $|f|,\ \mathrm{Re}(f),\ \mathrm{Im}(f) \in E$ for every $f \in E$, and
4. $U_{T_\gamma} f \in E$ for every $f \in E$ and $\gamma \in \Gamma$.

Definition 0.3. Let H be a Hilbert space. $\mathscr{L}(H)$ is the space of all bounded linear operators from H to H. [1]

A family $\mathcal{S} \subseteq \mathscr{L}(H)$ is a *semigroup (of operators)* if $UV \in \mathcal{S}$ for all $U, V \in \mathcal{S}$. It is a *contraction semigroup* if, in addition, $||U|| \leq 1$ for all $U \in \mathcal{S}$.

We call

$$\mathrm{fix}(\mathcal{S}) := \bigcap_{U \in \mathcal{S}} \mathrm{fix}(U) = \{f \in H \mid Uf = f \text{ for every } U \in \mathcal{S}\}$$

the *fixed space* of $\mathcal{S}$.[2][3]

[1] Note, $\mathscr{U}(H) \subseteq \mathscr{L}(H)$ and $||U|| = 1$ when $U \in \mathscr{U}(H)$, so $\mathscr{U}(H)$ is a contraction semigroup.

[2] $\mathrm{fix}(\mathcal{S}) \subseteq H_{\mathrm{ds}}$.

[3] Meeting Notes: 0-dimensional. Next most simple case would be scaling (1-dimensional), and then rotations on a place (2-dimensional).

The *closed convex hull* $\overline{\text{co}}\ A$ is the closure of the set of all convex combinations of elements of A.

The *closed linear hull* $\overline{\text{lin}}\ A$ is the closure of the set of all linear combinations of elements of A.

Definition 0.4. Let $U : \Gamma \to \mathscr{U}(H)$ be a unitary representation of a discrete abelian amenable group Γ.

A group homomorphism $\chi : \Gamma \to \mathbb{T}$, where $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$, is called a *character*. The *dual group* Γ^* of Γ is the set of all such characters equipped with the multiplication given by $(\chi_1\chi_2)(\gamma) := \chi_1(\gamma)\chi_2(\gamma)$ for $\gamma \in \Gamma$ and $\chi_1, \chi_2 \in \Gamma^*$.[4]

Definition 0.5. Let

$$M_f := \text{lin}\{U_\gamma f \mid \gamma \in \Gamma\}.$$

We call a subset $M \subseteq H$ an *invariant finite-dimensional subspace* if M_f is finite-dimensional and $U_\gamma f \in M$ for all $f \in M$ and $\gamma \in \Gamma$.

The closure

$$H_{\text{ds}} = \overline{\bigcup\{M \subseteq H \mid M \text{ invariant finite-dimensional subspace}\}} \subseteq H$$

is called the *discrete spectrum part* of U.[5]

Proposition 0.1 (cf. Jamneshan and Kreidler, 2025, Proposition 5.1.10)**.** *The eigenspaces ker*$(\chi - U)$ *for* $\chi \in \Gamma^*$ *are pairwise orthogonal. For* $M \subseteq H$*, then the following are equivalent:*

1. *M is an irreducible invariant finite-dimensional subspace.*
2. *M is an invariant linear subspace which is at most one-dimensional.*
3. *There is $\chi \in \Gamma^*$ and $f \in$ ker$(\chi - U)$, such that $M = \mathbb{C} \cdot f$.*

Definition 0.6. A unitary representation $U : \Gamma \to \mathscr{U}(H)$ of a topological group Γ is *strongly continuous* if $\Gamma \to H$, $x \mapsto U_x f$ is continuous for every $f \in H$.

Definition 0.7. For every discrete group Γ, the maps

$$L : \Gamma \to \mathscr{U}(\mathrm{L}^2(\Gamma)), \qquad x \mapsto L_x \tag{1}$$

$$R : \Gamma \to \mathscr{U}(\mathrm{L}^2(\Gamma)), \qquad x \mapsto R_x \tag{2}$$

with $L_x f := f \circ l_{x^{-1}}$ and $R_x f := f \circ r_{x^{-1}}$ for $f \in \mathrm{L}^2(\Gamma)$ and $x \in \Gamma$ are strongly continuous unitary representations. We call L and R the *left* and *right regular representation* of Γ, respectively.[6]

[4] Commutator subgroup and abelianisation?

[5] Meeting Notes: In the $\Gamma = \mathbb{Z}$ case, then H_{ds} can be broken up into one-dimensional objects that corresponds to an eigenfunction.

[6] As Γ is discrete, then any map $\Gamma \to H$ is continuous for any topological space H. Thus, $\Gamma \to \mathrm{L}^2(\Gamma)$ defined by $\gamma \mapsto L_\gamma f$ or $\gamma \mapsto R_\gamma f$ is always continuous. Hence, L and R are strongly continuous unitary representations.

Assume that $U : \Gamma \to \mathcal{U}(H)$ is a strongly continuous unitary representation of a discrete group Γ. Let further P be the orthogonal projection onto the fixed space $\mathrm{fix}(U(\Gamma))$. Then

$$(Pf \mid g) = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{\gamma\in\Phi_N} (U_\gamma f \mid g)$$

for all $f, g \in H$.

If $U : \Gamma \to \mathcal{U}(H)$ is a strongly continuous unitary representation of a compact group Γ, then U has discrete spectrum.[7]

Exercise 0.1. Show that

$$H = H_{\mathrm{ds}} \oplus H_{\mathrm{wm}}$$

and what happens to H_{ds} and H_{wm} when it is averaged by the Abstract Mean Ergodic Theorem for amenable groups.[8]

Proposition 0.2. *Let (X, T) be a measure-preserving system. Then $L^2(X)_{ds}$ is an invariant Markov sublattice of $L^2(X)$.*

Exercise 0.2 (Jamneshan and Kreidler, 2025, Lemma 6.1.12)**.** Assume that $U : \Gamma \to \mathcal{U}(H)$ is a strongly continuous unitary representation of a discrete group Γ. Let further P be the orthogonal projection onto the fixed space $\mathrm{fix}(U(\Gamma))$. Then $(Pf \mid g) = \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{\gamma\in\Phi_N} (U_\gamma f \mid g)$ for all $f, g \in H$.

Jamneshan, A. and Kreidler, H. (2025). '*ISem 28: Ergodic structure theory and applications*',.

[7]This Γ would be the compact group behind the factor subsystem of the larger system. For example, $\Gamma = \mathbb{Z}$ for the irrational α group rotation would give us compact $G = \{z \in \mathbb{C} \mid |z| = 1\}$. The homomorphism from $\Gamma \to \Gamma$ is $n \mapsto e^{2\pi i\alpha n}$.

[8](Proposition 5.3.3)

$$H_{\mathrm{wm}} = \left\{ f \in H \mid \lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{\varphi\in\Phi_N} |(U_\varphi f \mid f)|^2 = 0 \right\}.$$