A Short Proof of a Generalised Conjecture of Erdős for Amenable Groups

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Abstract

We follow steps provided by the paper published by (Host, 2019), 'A Short Proof of a Conjecture of Erdős Proved by Moreira, Richter and Robertson', as well as using the results provided by (Kra et al., 2022), to generalise the proof of Erdős's conjecture for amenable groups. The main result that we aim to prove is: 'every positively dense subset of an amenable group contains the group sumset of k infinite sets for every natural number k'.

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Notation

Term	Description
ℕ	The Natural Numbers: 1, 2, 3, … (Source)

1 Introduction

Theorem 1.1 (cf. (Host, 2019), Theorem 1)**.** *Let* $(\Gamma, \cdot)$ *be an amenable group. If* $A \subseteq \Gamma$ *has positive density, then there exists infinite subsets* B *and* C *of* Γ *such that* $B \cdot C \subset A$*.*

Theorem 1.2 (cf. (Host, 2019), Proposition 2)**.** *There exists a set of positive density not containing any sumset of positive density and an infinite set.*

Theorem 1.3 (cf. (Kra et al., 2022), Theorem 1.1)**.** *Let* $(\Gamma, \cdot)$ *be an amenable group. If* $A \subseteq \Gamma$ *has positive density then, for every* $k \in \mathbb{N}$*, there are infinite subsets* $B_1, ..., B_k \subset \Gamma$ *such that* $B_1 \cdot \dots \cdot B_k \subset A$*.*

2 Preliminaries

We will use $\mathbb{N} = \{1, 2, 3, ...\}$ and e to denote the identity of the group Γ.

2.1 Amenable Groups and Actions

Throughout this paper, unless otherwise specified, we will let $(\Gamma, \cdot)$ be a second countable discrete group. This also means that the Haar measure of Γ is the counting measure.

Definition 2.1. We define a *right-Følner sequence* in Γ as a sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ of finite subsets of Γ satisfying

$$\lim_{N\to\infty} \frac{\lambda(\Phi_N \cdot \gamma^{-1}) \cdot \Phi_N}{\lambda \Phi_N} = 1,$$

for all $\gamma \in \Gamma$.

Similarly, we define a *left-Følner sequence* in Γ as a sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ of finite subsets of Γ satisfying

$$\lim_{N\to\infty} \frac{\lambda(\gamma^{-1} \cdot \Phi_N) \cap \Phi_N}{\lambda \Phi_N} = 1,$$

for all $\gamma \in \Gamma$.

We call a sequence in Γ a *Følner sequence* if it is both a left and right Følner sequence.

For simplicity of this paper, we will use the alternative and equivalent definition that a group, Γ, is *amenable* (and second countable) if and only if it has a Følner sequence.

Definition 2.2 (Lindenstrauss (2001), Definition 1.1, Proposition 1.4)**.** A sequence of sets $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ will be said to be *tempered* if, for some $b > 0$ and all $n \in \mathbb{N}$,

$$\lambda \bigcup_{1\leq k<N} \Phi_k^{-1}\Phi_N \leq b\lambda\Phi_N. \tag{1}$$

is referred to as the *Shulman condition*.

1. Every Følner sequence $\Phi = (\Phi_N)_{N\in\mathbb{N}}$ has a tempered subsequence.
2. Every amenable group has a tempered Følner sequence.

Definition 2.3. For $A \subseteq \Gamma$ and Følner sequence $\Phi = (\Phi_N)_{n=1}^{\infty}$, we write

$$\overline{d}_\Phi(A) = \limsup_{N\to\infty} \frac{\lambda A \cap \Phi_N}{\lambda\Phi_N}$$
$$\underline{d}_\Phi(A) = \liminf_{N\to\infty} \frac{\lambda A \cap \Phi_N}{\lambda\Phi_N}$$

to be the *upper and lower densities of A with respect to* Φ, respectively. If these agree, then we can write

$$d_\Phi(A) = \lim_{N\to\infty} \frac{\lambda A \cap \Phi_N}{\lambda\Phi_N}$$

to be the *density of A with respect to* Φ. We also define the *upper Banach density of A* by

$$\overline{d}(A) = \sup d_\Phi(A) : \text{for Følner sequences } \Phi \text{ where } d_\Phi(A) \text{ exists.}$$

2.2 Topological Dynamics of Group Actions

Definition 2.4 (Bekka and Mayer (2000) Section 2)**.** An *action* of a group, Γ, on a measurable space $(X, \mathscr{B})$ is a measurable mapping

$$\Gamma \times X \to X,\ (\gamma, x) \mapsto \gamma.x$$

with the following properties:

1. Associativity: For all $\gamma, \gamma' \in \Gamma, x \in X$, then $\gamma.(\gamma'.x) = (\gamma \cdot \gamma').x$
2. Identity: There exists an identity element $e \in \Gamma$ such that $e.x = x$ for all $x \in X$.

3. Quasi-Invariance: For any $B \in \mathscr{B}$ and for all $\gamma \in \Gamma$, we have $\mu(\gamma.B) = 0$ if and only if $\mu(B) = 0$.

The action of Γ is also ergodic if it satisfies the additional property:

4. If $B \in \mathscr{B}$ and $\mu(B) = \mu(\gamma.B)$ for any $\gamma \in \Gamma$, then $\mu(B) = 0$ or $\mu(X \setminus B) = 0$.

Definition 2.5. A *topological dynamical system under the action of* Γ, denoted (X, Γ), is a compact metric space X that has continuous surjective maps, $(\gamma, x) \mapsto \gamma.x$, for all $\gamma \in \Gamma$.

Definition 2.6. Let $x \in X$, $\Phi = (\Phi_N)_{N \in \mathbb{N}}$ be a Følner sequence in Γ and μ a probability measure on X. Where δ_x is the Dirac mass at x, if

$$\frac{1}{|\Phi_N|} \sum_{\gamma \in \Phi_N} \delta_{\gamma.x} \underset{\text{weakly*}}{\longrightarrow} \mu \text{ as } N \to \infty,$$

then we say x *is generic for* μ *with respect to* Φ and we denote this with $x \in \text{gen}(\mu, \Phi)$.[1]

2.3 Recurrence Results & Ergodic Theorems

Theorem 2.1 (Bergelson (1985), Theorem 1.1)**.** *Let* $(X, \mathscr{B}, \mu)$ *be a probability space and suppose that* $B_n \in \mathscr{B}$ *such that* $\mu(B_n) = b > 0$ *for all* $n \in \mathbb{N}$*.*

Then there exists a positively dense index set $I \subset \mathbb{N}$ *such that, for any finite subset* $F \subseteq I$*, we have*

$$\mu\left(\bigcap_{i \in F} B_i\right) > 0.$$

Theorem 2.2 ({{< term-title ref= "thm-GeneralisedPET" removeURLs=true >}})**.**

2.4 Erdős Cubes & Cubic Measures

2.5 Factor Maps

2.6 Key Dynamical Results

Theorem 2.3 (cf. Host (2019), Theorem 3)**.** *Let* (X, Γ) *be a topological dynamical system where* Γ *is an amenable group,* $x_0 \in X$*,* E *be a clopen subset of* X *and*

$$A = \{\gamma \in \Gamma : \gamma.x_0 \in E\}.$$

Let $X \times X$ *be acted upon by* $\Gamma \times \Gamma$*. Let* $x_1 \in X$ *and* ν *be a measure on* $X \times X$ *such that* (x_0, x_1) *is generic along some Følner sequence* $\Phi = (\Phi_N)_{N \in \mathbb{N}}$*.*

[1] consider tempered separately as the FCP construction only depends on sequential compactness

Assume there exists $\varepsilon > 0$ and a sequence $(s_i)_{i\in\mathbb{N}}$ in Γ such that

$$s_i.x_0 \to x_1 \text{ as } i \to \infty \text{ and } \nu(s_i^{-1}.E \times E) \geq \varepsilon \text{ for all } i.$$

Then there exist infinite subsets $B, C \subseteq \Gamma$ such that $B \cdot C \subset A$.

Theorem 2.4 (cf. Kra et al. (2022), Theorem 3.5)**.** *Let (X, μ, Γ) be an ergodic system under the action of Γ and let $E \subset X$ be an open set with $\mu(E) > 0$.*

If $a \in X$ is generic for μ with respect to some Følner sequence Φ then, for every $k \in \mathbb{N}$, there exists 2-dimensional Erd"{o}s cube $\underline{x} = (x_{00}, x_{01}, x_{10}, x_{11}) \in X^{[[2]]}$ with $x_{00} = a$ and $x_{11} \in E$.

Theorem 2.5 (cf. Host (2019), Theorem 4.4)**.** *Let (X, μ, Γ) be an ergodic system under the action of Γ and let $E \subset X$ be an open set with $\mu(E) > 0$.*

If $a \in X$ is generic for μ with respect to some Følner sequence Φ then, for every $k \in \mathbb{N}$, there exists k-dimensional Erd"{o}s cubes $\underline{x} \in X^{[[k]]}$ with $x_{\vec{0}} = a$ and $x_{\vec{1}} \in E$.

3 Proof of Theorem Theorem 2.4

3.1 Furstenberg's Correspondence Princple

Theorem 3.1 (cf. Kra et al. (2022), Theorem 2.10)**.** *Let Γ be an amenable group, $A \subset \Gamma$, and Φ be a Følner sequence in Γ such that the limit*

$$\delta = \lim_{N\to\infty} \frac{\lambda A \cap \Phi_N}{\lambda \Phi_N}$$

exists.

Then there exists an ergodic system (X, μ, Γ) that is acted on by Γ, a clopen set $E \subset X$, a Følner sequence Ψ in Γ, and a point $a \in X$ that is generic with μ with respect to Ψ such that $\mu(E) \geq \delta$ and

$$A = \gamma \in \Gamma : \gamma.a \in E.$$

3.2 Kronecker Factor

Proposition 3.1 ({{< term-title ref= "prp-KroneckerGeneric" removeURLs=true >}})**.**

3.3 Choosing a point x_1

3.4 The joining ν

3.5 Proof Conclusion

4 Proof of Corollary (ref)

5 Proof of Theorem (ref)

Discussion

Host, B. (2019). '*A short proof of a conjecture of erdös proved by moreira, richter and robertson*', Available at: https://arxiv.org/abs/1904.09952.

Kra, B., et al. (2022). '*Infinite sumsets in sets with positive density*', Available at: https://arxiv.org/abs/2206.01786.

Lindenstrauss, E. (2001). 'Pointwise theorems for amenable groups', Inventiones mathematicae*,* 146 (2), pp. 259–295. https://doi.org/10.1007/s002220100162.

Bekka, M. B. and Mayer, M. (2000). *Ergodic theory and topological dynamics of group actions on homogeneous spaces.* Cambridge University Press. https://doi.org/10.1017/cbo9780511758898.

Bergelson, V. (1985). 'Sets of recurrence of zm-actions and properties of sets of differences in zm', Journal of the London Mathematical Society*,* s2-31 (2), pp. 295–304. https://doi.org/10.1112/jlms/s2-31.2.295.